

**OPEN ACCESS**

# A Newtonian problem as an insightful tool for the behavior of gravitational-wave sources

To cite this article: Theocharis A Apostolatos *et al* 2013 *J. Phys.: Conf. Ser.* **453** 012001

View the [article online](#) for updates and enhancements.

## Related content

- [Local interior estimates for the Euler-Einstein system](#)  
Taxiarchis Papakostas and Demetrios Pliakis
- [Searching for non-Kerr objects](#)  
Theocharis A Apostolatos, Georgios Lukes-Gerakopoulos and George Contopoulos
- [Mind the Resonances: Final stages of accretion into bumpy black holes](#)  
Georgios Lukes-Gerakopoulos and George Contopoulos



**IOP | ebooks™**

Bringing together innovative digital publishing with leading authors from the global scientific community.

Start exploring the collection—download the first chapter of every title for free.

# A Newtonian problem as an insightful tool for the behavior of gravitational-wave sources

Theocharis A. Apostolatos<sup>1†</sup>, George Pappas<sup>1,2</sup> and Katerina Chatziioannou<sup>3</sup>

<sup>1</sup> Section of Astrophysics, Astronomy, and Mechanics, Department of Physics, University of Athens, Panepistimiopolis Zografos GR15783, Athens, Greece

<sup>2</sup> Theoretical Astrophysics, IAAT, Eberhard Karls University of Tübingen, Tübingen 72076, Germany

<sup>3</sup> Department of Physics, Montana State University, Bozeman, MT 59717, USA.

E-mail: † [thapostol@phys.uoa.gr](mailto:thapostol@phys.uoa.gr)

**Abstract.** We present the intriguing mathematical and physical similarities arising from the study of the Kerr metric and the Euler’s problem of two fixed gravitational centers. We show how one could extend these similarities to physical problems that emerge from the above integrable problems, when both are slightly perturbed.

## 1. Introduction

There is an old problem of classical mechanics which is characterized by a lot of properties that are similar to those of an important astrophysical problem; that of a Kerr black hole. That problem is the so called Euler’s problem of two fixed gravitational centers. Although the two problems do not seem to describe analogous physical configurations (moreover the two problems arise in a completely different theoretical framework), the similarities between the two problems are so deep that not only one can identify a number of quantitative and qualitative correspondences between the two, but one can use one of them to get insight about the other. Especially the newtonian problem could be used as a simple Newtonian analogue of a Kerr black hole in order to investigate the behavior of the latter one when this is somehow perturbed.

In Section 2 we give a short description of the Euler’s problem. In Section 3 we present a list of all similarities between the two physical problems. In Section 4 we show how one could exploit this intriguing similarity between the two problems to get insight about one of them by performing numerical simulations with the other one.

## 2. The Euler’s problem

The Newtonian gravitational field of two stationary gravitating point masses  $m_1, m_2$  located at a fixed distance  $2a$  from each other, one at  $a\hat{\mathbf{z}}$  and the other at  $-a\hat{\mathbf{z}}$  is given by

$$V(\mathbf{r}) = -\frac{Gm_1}{r_1} - \frac{Gm_2}{r_2} = -\frac{Gm_1}{|\mathbf{r} - a\hat{\mathbf{z}}|} - \frac{Gm_2}{|\mathbf{r} + a\hat{\mathbf{z}}|}, \quad (1)$$



where  $r_1, r_2$  are the distances between the two masses and the point -located at  $\mathbf{r}$ - where the potential is measured. This problem is better analyzed in spheroidal coordinates:

$$\xi \equiv \frac{r_1 + r_2}{2a}, \eta \equiv \frac{r_1 - r_2}{2a}. \quad (2)$$

By construction this problem actually describes a prolate (along the  $z$ -axis) configuration, while the gravitational field of a Kerr black hole is oblate due to the black-hole's spin. Moreover if the two masses in the Euler's problem are not equal, the gravitational field is not reflection symmetric in contrary to what happens with a Kerr black hole. The above discrepancies between the two problems can be easily fixed by replacing  $a$  by an imaginary distance  $ia$  and setting  $m_1 = m_2 = M/2$  in the Euler's problem. Although the replacement of a distance by an imaginary one cannot be actually realized by a physical nonsingular distribution of matter, the corresponding potential is perfectly real-valued (when  $m_1 = m_2$ ) and can be easily studied.

The actual form of the potential in the latter case -which is the case that we are going to study as the Newtonian analogue of Kerr metric- assumes the following form in spherical and spheroidal coordinates:

$$V = -\frac{GM}{\sqrt{2}R^2} \sqrt{R^2 + r^2 - a^2} = -\frac{GM\xi}{a(\xi^2 + \eta^2)}, \quad (3)$$

where

$$R^2 = \sqrt{(r^2 - a^2)^2 + (2a\mathbf{r} \cdot \hat{\mathbf{z}})^2}. \quad (4)$$

The complicated form in spherical coordinates arises from the summation of two complex conjugate quantities (cf. Eq. 1). It should be noted that the quantity under the square root is always positive as one could easily verify by a simple inspection of the form of  $R$ . It only becomes zero on the equatorial  $\mathbf{r} \cdot \hat{\mathbf{z}} = 0$  circle  $r = a$ .

Henceforth, we will use geometrized units  $G = c = 1$  for the Newtonian problem so that it is easier to compare with the relativistic one.

### 3. Similarities

- (i) Both the Euler's problem (henceforth the Euler's problem will refer to the gravitational field of two equal masses at an imaginary distance  $2ia$  apart, as it is described in Eq. (3)) and the Kerr black hole are gravitational fields in which the equations of motion (either Newtonian or relativistic) of an orbiting test body are integrable. This is a very important similarity that renders both problems separable and thus easily solvable by the Hamilton-Jacobi technique.
- (ii) Both problems describe oblate fields in which the geodesic orbits are precessing around the equatorial plane. Thus the spherical azimuthal angle  $\theta$  of a point test particle performs oscillations around  $\pi/2$ .
- (iii) Apart from the obvious symmetries (stationarity and axisymmetry) of the two problems that lead to the conservation of the energy  $E$  and the  $z$ -angular momentum  $L_z$ , there is an extra conserved quantity in both cases: The third integral  $b$  in the Euler's problem (see [1]) and the Carter constant  $Q$  for the Kerr metric. Both integrals of motion are quadratic with respect to momenta. However the analogy between the two quantities is not only restricted to general qualitative characteristics. The Carter constant can be written as a function of  $\theta$  and  $\theta$ -momentum ( $p_\theta$ ) as

$$Q = p_\theta^2 + \cos^2 \theta \left[ a^2(1 - E^2) + L_z^2 / \sin^2 \theta \right]. \quad (5)$$

If we make the following replacement of the spheroidal coordinate  $\eta$  of the Euler's problem  $\eta \rightarrow \cos \theta$ , then  $p_\eta^2(1 - \eta^2) \rightarrow p_\theta^2$  (the latter is a direct consequence of the former one), and the Newtonian approximation of the relativistic energy  $E^2 \rightarrow 1 + 2E_N$  where  $E_N$  is the conserved Newtonian energy of the Euler's problem then the third integral  $b$  assumes exactly the same form as  $Q$  (apart from some additive terms related to the rest conserved quantities):

$$b = -Q - L_z^2 - 2a^2 E_N. \quad (6)$$

The same expression connecting  $b$  and  $Q$ , could be shown to hold true also when the two conserved quantities are written as functions of  $r$  and  $p_r$  (for  $Q$ ), and  $r = \xi a$  and  $p_r = p_\xi/a$  (for  $b$  respectively). The only difference here is that besides the correspondence between  $E_N$  and  $E$  mentioned above suitable approximations of various functions of  $r$  should be made.

- (iv) Will showed [2] that the only axisymmetric and reflection symmetric Newtonian potential that leads to a conserved quantity which is bilinear with respect to momenta is the original Euler's problem (the one with a real distance  $a$ ) that has exactly the same spectrum of mass multipole moments as the Kerr one apart from a sign; namely two consecutive non-vanishing mass moments of the initial Euler's problem satisfy the equation

$$M_{2l} = a^2 M_{2l-2}, \quad (7)$$

with  $M_0 = M$ . (Due to reflection symmetry only the even multipoles are non-zero.) If  $a$  is replaced by  $ia$  in the original Euler's problem one gets back the exact multipolar structure of a Kerr black hole; that is

$$M_{2l} = -a^2 M_{2l-2}^1 \quad (8)$$

with  $M_0 = M$ . This particular similarity manifests more clearly the physical similarity between the two fields.

- (v) Usually a Newtonian gravitational potential (like the monopole one) does not have something analogous to an innermost stable circular orbit (ISCO). However the Euler's problem actually possesses an ISCO. The radius of ISCO for the Euler's problem is

$$\xi_{\text{ISCO}} = \sqrt{3}. \quad (9)$$

It is rather clumsy to compare the corresponding radius  $r_{\text{ISCO}} = a\xi_{\text{ISCO}}$  with the  $a$  dependence of the radius of ISCO in Kerr, since there are two such radii; one for retrograde orbits and one for prograde ones. Here we simply note the existence of ISCO in the Newtonian problem as an analogy with orbits in Kerr.

- (vi) Since both problems (the Newtonian and the relativistic) are integrable all bound orbits are characterized by 3 fundamental frequencies related to the  $r$ -,  $\theta$ -oscillations and  $\phi$ -libration. Actually these are the frequencies that one could measure through gravitational waves emitted from small objects orbiting around massive Kerr black holes. The behavior of these frequencies, though not analytically equal between the two problems, they show some analogies. For example for both problems  $\omega_r/\omega_\theta \rightarrow 1$  for large radii. Also for a large range of parameters ( $a$ , and inclination angle,  $\iota$ , of the orbit) the ratios of frequencies have qualitatively similar behavior as a function of orbital semi-latus rectum. Finally the resonance condition  $\omega_r/\omega_\theta = 2$  was never met in all cases we examined for both cases. The avoidance of such a resonance is quite essential to keep orbits that start almost circular ( $r = \text{const}$ ) to remain circular after an adiabatic evolution of the orbit (see [3]).

<sup>1</sup> Here the comparison is made only between the mass moments of the two gravitational fields. The analogue of the current-mass moments in relativity does not affect the gravitational field in Newtonian gravity.

- (vii) In a recent paper [4] it has been shown that the wave equation in a Newtonian gravitational potential becomes separable only for the monopole potential and the Euler problem (either the prolate or the oblate one). It is quite interesting that the oblate Euler's problem leads to a set of differential equations that resembles a lot the corresponding wave equations that describe propagation of scalar waves in a Kerr background. To be more specific the azimuthal part is exactly the same for the two problems, while the radial part, though not analytically equal, they both have similar properties especially for some particular values of the parameters of the Newtonian wave equation.

#### 4. Exploiting the analogies

The most interesting similarity though between the two problems is that when their integrability is slightly destroyed (for example by constructing exact solutions of vacuum Einstein equations that are parametrized by a single parameter which measures deviations from a Kerr metric, and adding a small point mass at the origin in the Euler's problem) both problems demonstrate very similar behavior. This is a generic outcome of the KAM theorem for all slightly non-integrable problems. Thus according to that theorem in the Poincaré sections of such problems chains of Birkhoff islands form near resonances (when the ratio of fundamental frequencies  $\omega_r/\omega_\theta$  is rational). The idea then is to exploit such similarities to check how a perturbed Kerr metric is expected to behave under gravitational radiation evolution by studying the adiabatic evolution of an orbit in the perturbed Euler's problem. In the Newtonian case it is easy to assume a suitably small dissipative force acting on the particle and study the behavior of the orbit when it passes a resonance. Is the corresponding crossing slower or faster than what one yields when the orbit is evolved by means of the average energy and angular momentum loss due to the force, where the average losses are computed outside the resonance? This is a very difficult problem to address in relativity since the radiation self-force is quite difficult to compute even in a pure Kerr background. In a generic perturbed Kerr background the computation of self-force seems rather intractable.

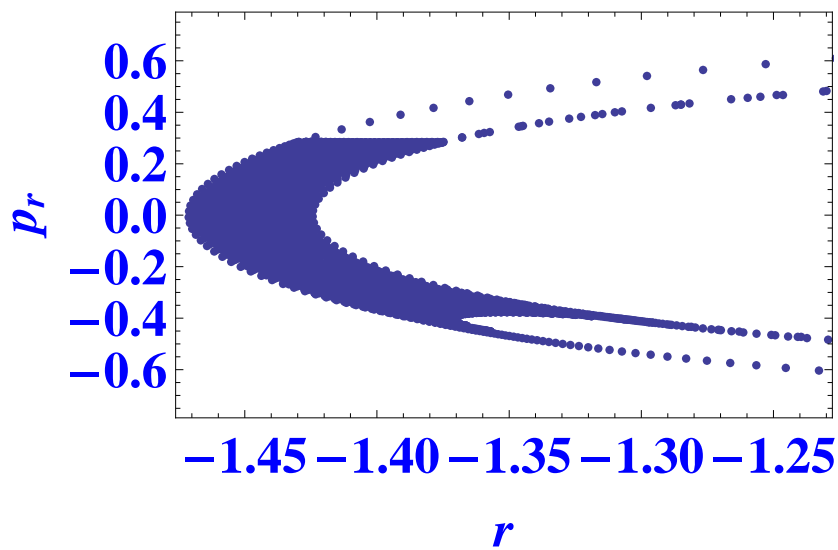
The first numerical runs show that the process of crossing a resonance island is actually rather slower than what one obtains from averaging the losses. If this is a general outcome, then the plateau effect suggested in [5, 6] might be further enhanced in perturbed Kerr metrics. Therefore by seeking such plateaus in gravitational-wave-data analysis, one could pursue exotic non-Kerr objects. More details will be presented in a future paper [7]

#### Acknowledgments

We would like to thank Kostas Kokkotas for the hospitality at the University of Tübingen. This work has been supported by the I.K.Y. (IKYDA 2010) and G.P. would also like to acknowledge DAAD scholarship number A/12/71258.

#### References

- [1] Landau L. D and Lifshitz E. M., 1976, *Mechanics* (Third Edition), Butterworth-Heinemann.
- [2] Will C., 2010, *Phys. Rev. Lett.* **102**, 061101
- [3] Kennefick D. and Ori A., 1996, *Phys. Rev. D* **53**, 4319
- [4] Glampedakis K. and Apostolatos T. A., submitted in *Class. & Quant. Gravity*
- [5] Apostolatos T. A., Lukes-Gerakopoulos G. and Contopoulos G., 2009, *Phys. Rev. Lett.* **103**, 111101
- [6] Lukes-Gerakopoulos G., Apostolatos T. A. and Contopoulos G., 2010, *Phys. Rev. D* **81**, 124005
- [7] T.A. Apostolatos, G. Pappas and K. Chatziioannou, paper in preparation



**Figure 1.** The Euler's problem with an added small mass at the origin was transformed to a slightly perturbed integrable problem that could be used to mimic a perturbed Kerr black hole. Here is plotted the Poincaré section  $\theta = \pi/2$  of a phase orbit that hits the resonance of  $2 : 3$ , due to adiabatic inspiral caused by an artificial dissipative force. The orbit spends a much longer time inside the resonance island, compared to the time one would expect based on the average loss of energy and angular momentum when the orbit was still outside the resonance region.